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# Spherical functions on the space of $\mathbb{P}$ -adic unitary hermitian matrices, the case of odd size (Automorphic Forms and Related Zeta Functions)

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# Spherical functions on the space of $p$ -adic unitary hermitian matrices, the case of odd size

by

Yumiko Hironaka and Yasushi Komori

## §0 Introduction

We have been interested in the harmonic analysis on  $p$ -adic homogeneous spaces based on spherical functions. We have considered the space of  $p$ -adic unitary hermitian matrices of even size in [HK], and in the present article we will report that of odd size. For details, please see the full version [HK2]. The results can be formulated in parallel, though the groups acting the spaces have different root structures,  $C_n$  for even and  $BC_n$  for odd. The spaces have a natural close relation to the theory of automorphic functions and classical theory of sesquilinear forms (e.g. [H3], [HS]).

We fix an unramified quadratic extension  $k'/k$  of  $p$ -adic field  $k$  such that  $2 \notin \mathfrak{p}$ , and consider hermitian and unitary matrices with respect to  $k'/k$ , and denote by  $a^*$  the conjugate transpose of  $a \in M_{mn}(k')$ . Let  $\pi$  be a prime element of  $k$  and  $q$  the cardinality of the residue class field  $\mathcal{O}_k/(\pi)$  and we normalize the absolute value on  $k$  by  $|\pi| = q^{-1}$ .

Denote by  $j_m \in GL_m(k)$  the matrix whose all anti-diagonal entries are 1 and others are 0. Set

$$\begin{aligned} G &= U(j_m) = \{g \in GL_m(k') \mid gj_m g^* = j_m\}, \quad K = G(\mathcal{O}_{k'}) \\ X &= \{x \in X \mid x^* = x, \Phi_{xj_m}(t) = \Phi_{j_m}(t)\}, \\ g \cdot x &= gxg^*, \quad (g \in G, x \in X), \end{aligned}$$

where  $\Phi_y(t)$  is the characteristic polynomial of matrix  $y$ . We note that  $X$  is a single  $G(\bar{k})$ -orbit over the algebraic closure  $\bar{k}$  of  $k$ . Set

$$n = \left\lfloor \frac{m}{2} \right\rfloor.$$

According to the parity of  $m$ ,  $G$  has the root structure of type  $C_n$  for even  $m$  and type  $BC_n$  for odd  $m$ . It is known in general that the spherical functions on various  $p$ -adic groups  $\Gamma$  can be expressed in terms of the specialization of Hall-Littlewood polynomials of the corresponding root structure of  $\Gamma$  (cf. [M2, §10], also [Car, Theorem 4.4]). For the present space  $X$ , the main term of spherical functions can be written by using Hall-Littlewood polynomials of type  $C_n$  with different specialization according to the parity of  $m$  (cf. Theorem 3 below).

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Let us note the results of even size in [HK] and odd size simultaneously, so that one can compare the results.

**Theorem 1** (1) *A set of complete representatives of  $K \backslash X$  can be taken as*

$$\{x_\lambda \mid \lambda \in \Lambda_n^+\}, \quad (0.1)$$

where

$$x_\lambda = \begin{cases} \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}) & \text{if } m = 2n \\ \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, 1, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}) & \text{if } m = 2n + 1, \end{cases}$$

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\},$$

(2) *There are precisely two  $G$ -orbits in  $X$  represented by  $x_0 = x_0 = 1_m$  and  $x_1 = x_{(1,0,\dots,0)}$ .*

The proof of Cartan decomposition for odd size needs a more delicate calculation than that for even size. If  $k$  has even residual characteristic, there are some  $K$ -orbits without any diagonal matrix besides the above types, independent of the parity of the size.

A spherical function on  $X$  is a  $K$ -invariant function on  $X$  which is a common eigenfunction with respect to the convolutive action of the Hecke algebra  $\mathcal{H}(G, K)$ , and a typical one is constructed by Poisson transform from relative invariants of a parabolic subgroup. We take the Borel subgroup  $B$  consisting of upper triangular matrices in  $G$ . For  $x \in X$  and  $s \in \mathbb{C}^n$ , we consider the following integral

$$\omega(x; s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i} dk, \quad (0.2)$$

where  $d_i(y)$  is the determinant of the lower right  $i$  by  $i$  block of  $y$ ,  $1 \leq i \leq n$ . Then the right hand side of (0.2) is absolutely convergent for  $\text{Re}(s_i) \geq 0$ ,  $1 \leq i \leq n$ , and continued to a rational function of  $q^{s_1}, \dots, q^{s_n}$ . Since  $d_i(x)$ 's are relative  $B$ -invariants on  $X$  such that

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)) \quad (p \in B, x \in X, 1 \leq i \leq n),$$

we see  $\omega(x; s)$  is a spherical function on  $X$  which satisfies

$$f * \omega(x; s) = \lambda_s(f) \omega(x; s), \quad f \in \mathcal{H}(G, K)$$

$$\lambda_s(f) = \int_B f(p) \prod_{i=1}^n |\psi_i(p)|^{-s_i} \delta(p) dp,$$

where  $dp$  is the left invariant measure on  $B$  with modulus character  $\delta$ . We introduce the new variable  $z \in \mathbb{C}^n$  related to  $s$  by

$$s_i = -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q}, \quad 1 \leq i \leq n-1$$

$$s_n = \begin{cases} -z_n - \frac{1}{2} + \frac{\pi\sqrt{-1}}{\log q} & \text{if } m = 2n \\ -z_n - 1 + \frac{\pi\sqrt{-1}}{2\log q} & \text{if } m = 2n + 1, \end{cases} \quad (0.3)$$

and denote  $\omega(x; z) = \omega(x; s)$  and  $\lambda_s = \lambda_z$ .

The Weyl group  $W$  of  $G$  relative to  $B$  acts on rational characters of  $B$ , hence on  $z$  and  $s$  also. The group  $W$  is generated by  $S_n$  which acts on  $z$  by permutation of indices and by  $\tau$  such that  $\tau(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, -z_n)$ . We will give the functional equation of  $\omega(x; z)$  with respect to  $W$ . To describe the results we prepare some notation. We set

$$\Sigma^+ = \Sigma_s^+ \sqcup \Sigma_\ell^+, \\ \Sigma_s^+ = \{e_i + e_j, e_i - e_j \mid 1 \leq i < j \leq n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\},$$

where  $e_i \in \mathbb{Z}^n$  is the  $i$ -th unit vector, and define a pairing

$$\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}, \quad \langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i.$$

**Theorem 2** (1) *For any  $\sigma \in W$ , one has*

$$\omega(x; z) = \prod_{\alpha} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}} \cdot \omega(x; \sigma(z)),$$

where  $\alpha$  runs over the set  $\{\alpha \in \Sigma_s^+ \mid -\sigma(\alpha) \in \Sigma^+\}$  for  $m = 2n$  and  $\{\alpha \in \Sigma^+ \mid -\sigma(\alpha) \in \Sigma^+\}$  for  $m = 2n + 1$ .

(2) *The function  $G(z) \cdot \omega(x; z)$  is holomorphic and  $W$ -invariant, hence belongs to  $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$ , where*

$$G(z) = \prod_{\alpha} \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle - 1}},$$

and  $\alpha$  runs over the set  $\Sigma_s^+$  for  $m = 2n$  and  $\Sigma^+$  for  $m = 2n + 1$ .

As for the explicit formula of  $\omega(x; z)$  it suffices to give for each  $x_\lambda$  by Theorem 1 (1).

**Theorem 3** (Explicit formula) *For each  $\lambda \in \Lambda_n^+$ , one has*

$$\omega(x_\lambda; z) = \frac{c_n}{G(z)} \cdot q^{\langle \lambda, z_0 \rangle} \cdot Q_\lambda(z; t),$$

where  $G(z)$  is given in Theorem 2,  $z_0$  is the value in  $z$ -variable corresponding to  $s = 0$ ,

$$c_n = \begin{cases} \frac{(1 - q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n \\ \frac{(1 + q^{-1})(1 - q^{-2})^n}{w_m(-q^{-1})} & \text{if } m = 2n + 1, \end{cases} \quad w_m(t) = \prod_{i=1}^m (1 - t^i),$$

$$Q_\lambda(z; t) = \sum_{\sigma \in W} \sigma(q^{-\langle \lambda, z \rangle} c(z; t)), \quad c(z; t) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}}$$

$$t_\alpha = \begin{cases} t_s & \text{if } \alpha \in \Sigma_s^+ \\ t_\ell & \text{if } \alpha \in \Sigma_\ell^+, \end{cases} \quad t_s = -q^{-1}, \quad t_\ell = \begin{cases} q^{-1} & \text{if } m = 2n \\ -q^{-2} & \text{if } m = 2n + 1. \end{cases}$$

We see  $Q_\lambda(z; t)$  belongs to  $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$  by Theorem 2. It is known that  $Q_\lambda(z; t) = W_\lambda(t)P_\lambda(z; t)$  with Hall-Littlewood polynomial  $P_\lambda(z; t)$  and Poincaré polynomial  $W_\lambda(t)$  and the set  $\{P_\lambda(z; t) \mid \lambda \in \Lambda_n^+\}$  forms an orthogonal  $\mathbb{C}$ -basis for  $\mathcal{R}$  for each  $t_\alpha \in \mathbb{R}, |t_\alpha| < 1$  (cf. [HK, Appendix C]). As is written as above, we need the different specialization for  $Q_\lambda(z; t)$  according to the parity of  $m$ , and  $G(z)$  is different also.

In particular, we have

$$\omega(x_0; z) = \frac{(1 - q^{-1})^n w_n(-q^{-1}) w_{m'}(-q^{-1})}{w_m(-q^{-1})} \cdot G(z)^{-1}, \quad m' = \left[\frac{m+1}{2}\right] \quad (0.4)$$

and we may modify the spherical function  $\omega(x; z)$  as

$$\Psi(x; z) = \frac{\omega(x; z)}{\omega(x_0; z)} \in \mathcal{R}. \quad (0.5)$$

We define the spherical Fourier transform on the Schwartz space  $\mathcal{S}(K \backslash X)$  as follows

$$\hat{\cdot}: \mathcal{S}(K \backslash X) \longrightarrow \mathcal{R}, \quad \varphi \longmapsto \hat{\varphi}(z) = \int_X \varphi(x) \Psi(x; z) dx$$

where  $dx$  is a  $G$ -invariant measure on  $X$ .

**Theorem 4** (1) *The above spherical transform is an  $\mathcal{H}(G, K)$ -module isomorphism and  $\mathcal{S}(K \backslash X)$  is a free  $\mathcal{H}(G, K)$ -module of rank  $2^n$ .*

(2) *All the spherical Fourier functions on  $X$  are parametrized by  $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\right)^n / W$  through  $\lambda_z$ , and the set  $\left\{ \Psi(x; z + u) \mid u \in \{0, \frac{\pi\sqrt{-1}}{\log q}\}^n \right\}$  forms a  $\mathbb{C}$ -basis of spherical functions corresponding to  $z$ .*

(3) *(Plancherel formula) Set a measure  $d\mu(z)$  on  $\mathfrak{a}^* = \left\{ \sqrt{-1} \left( \mathbb{R}/\frac{2\pi}{\log q} \mathbb{Z} \right) \right\}^n$  by*

$$d\mu(z) = \frac{1}{2^n n!} \cdot \frac{w_n(-q^{-1}) w_{m'}(-q^{-1})}{(1 + q^{-1})^{m'}} \cdot \frac{1}{|c(z; t)|^2} dz, \quad m' = \left[\frac{m+1}{2}\right]$$

where  $dz$  is the Haar measure on  $\mathfrak{a}^*$ . By an explicitly given normalization of  $dx$  depending on the parity of  $m$ , one has

$$\int_X \varphi(x) \overline{\psi(x)} dx = \int_{\mathfrak{a}^*} \hat{\varphi}(z) \overline{\hat{\psi}(z)} d\mu(z) \quad (\varphi, \psi \in \mathcal{S}(K \backslash X)).$$

(4) *(Inversion formula) For any  $\varphi \in \mathcal{S}(K \backslash X)$ , one has*

$$\varphi(x) = \int_{\mathfrak{a}^*} \hat{\varphi}(z) \Psi(x; z) d\mu(z), \quad x \in X.$$

## §1 The Space $X$ of unitary hermitian matrices

Let  $k'$  be an unramified quadratic extension of a  $\mathfrak{p}$ -adic field  $k$  of odd residual characteristic and consider hermitian and unitary matrices with respect to  $k'/k$ , and denote by  $a^*$  the conjugate transpose of  $a \in M_{mn}(k')$ . Let  $\pi$  be a prime element of  $k$  and  $q$  the cardinality of the residue class field  $\mathcal{O}_k/(\pi)$  and we normalize the absolute value on  $k$  by  $|\pi| = q^{-1}$  and denote by  $v_\pi(\cdot)$  the additive value on  $k$ . We fix a unit  $\epsilon \in \mathcal{O}_k^\times$  for which  $k' = k(\sqrt{\epsilon})$ .

We consider the unitary group

$$G = G_n = \{g \in GL_{2n+1}(k') \mid g^* j_{2n+1} g = j_{2n+1}\}, \quad j_{2n+1} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in M_{2n+1},$$

and the space  $X$  of unitary hermitian matrices in  $G$

$$X = X_n = \{x \in G \mid x^* = x, \Phi_{xj_{2n+1}}(t) = (t^2 - 1)^n(t - 1)\}, \quad (1.1)$$

where  $\Phi_y(t)$  is the characteristic polynomial of the matrix  $y$ . It should be noted that (1.1) implies  $\det x = 1$ . We note that  $X$  is a single  $G(\bar{k})$ -orbit containing  $1_{2n+1}$  over the algebraic closure  $\bar{k}$  of  $k$  ([HK, Appendix A]). The group  $G$  acts on  $X$  by

$$g \cdot x = gxg^* = gxj_{2n+1}g^{-1}j_{2n+1}, \quad g \in G, x \in X.$$

We fix a maximal compact subgroup  $K$  of  $G$  by

$$K = K_n = G \cap M_{2n+1}(\mathcal{O}_{k'}),$$

(cf. [Sa, §9]), and take a Borel subgroup  $B$  of  $G$ , which consists of all the upper triangular matrices in  $G$  and is given by

$$B = \left\{ \begin{pmatrix} A & & \\ & u & \\ & & j_n A^{*-1} j_n \end{pmatrix} \begin{pmatrix} 1_n & \beta & C \\ & 1 & -\beta^* j_n \\ & & 1_n \end{pmatrix} \mid \begin{array}{l} A \in B_n, \beta \in (k')^n, u \in \mathcal{O}_{k'}^\times, C \in M_n(k') \\ \beta\beta^* + Cj_n + j_n C^* = 0_n \end{array} \right\}, \quad (1.2)$$

where  $B_n$  is the set of all the upper triangular matrices in  $GL_n(k')$ ,  $\mathcal{O}_{k'}^\times = \{u \in \mathcal{O}_{k'}^\times \mid N(u) = 1\}$ . Here and hereafter empty entries in matrices should be understood as 0 and  $N$  as the norm map  $N_{k'/k}$ . The group  $G$  satisfies the Iwasawa decomposition  $G = BK = KB$ .

In this section, we give the  $K$ -orbit decomposition and the  $G$ -orbit decomposition of the space  $X$ .

**Theorem 1.1** *The  $K$ -orbit decomposition of  $X_n$  is given as follows:*

$$X_n = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda, \quad (1.3)$$

where

$$\begin{aligned} \Lambda_n^+ &= \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}, \\ x_\lambda &= \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, 1, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}). \end{aligned}$$

We recall the case of unramified hermitian matrices. The group  $GL_m(k')$  acts on the set  $\mathcal{H}_m(k') = \{x \in GL_m(k') \mid x^* = x\}$  by  $g \cdot x = gxg^*$ , and it is known (cf. [Ja])

$$\mathcal{H}_m(k') = \bigsqcup_{\lambda \in \Lambda_m} GL_m(\mathcal{O}_{k'}) \cdot \pi^\mu \quad (1.4)$$

where

$$\Lambda_m = \{\mu \in \mathbb{Z}^m \mid \mu_1 \geq \dots \geq \mu_m\}, \quad \pi^\mu = \text{Diag}(\pi^{\mu_1}, \dots, \pi^{\mu_m}).$$

Hence, we see that  $K \cdot x_\lambda \cap K \cdot x_\mu = \emptyset$  if  $\lambda \neq \mu$  in  $\Lambda_n^+$ . Moreover, since  $N(\mathcal{O}_{k'}^\times) = \mathcal{O}_k^\times$ , we see that any diagonal  $x \in X$  is reduced to some  $x_\lambda$ ,  $\lambda \in \Lambda_n^+$  by the action of  $K$ .

The strategy of the proof of Theorem 1.1 is the same as in the even case [HK]; general  $n$  cases are reduced to the case  $n = 1$ . However the case  $n = 1$ , the size of matrices is 3, is more complicated and technical than the even case.

**Remark 1.2** *If  $k$  has even residual characteristic, then there are some  $K$ -orbits without any diagonal matrix. In fact, the following matrices are contained in  $X_n$  and can not be diagonalized by the action of  $K$ , for  $n = r + s$ ,  $s > 0$  :*

$$\begin{pmatrix} 1_r & & & \\ & & -j_{\frac{s}{2}} & \\ & j_{s+1} & & \\ & -j_{\frac{s}{2}} & & \\ & & & 1_r \end{pmatrix} \quad (\text{if } 2 \mid s), \quad \begin{pmatrix} 1_r & & & \\ & & j_{\frac{s+1}{2}} & \\ & & -j_s & \\ & j_{\frac{s+1}{2}} & & \\ & & & 1_r \end{pmatrix} \quad (\text{if } 2 \nmid s).$$

Before giving the  $G$ -orbit decomposition of  $X_n$ , we recall the case of unramified hermitian matrices. It is known that there are precisely two  $GL_m(k')$ -orbits in  $\mathcal{H}_m(k')$  for  $m \geq 1$ :

$$\begin{aligned} \mathcal{H}_m(k') &= GL_m(k') \cdot 1_m \sqcup GL_m(k') \cdot \pi^{(1,0,\dots,0)}, \\ &= \left( \bigsqcup_{\substack{\mu \in \Lambda_m \\ |\mu| \text{ is even}}} GL_m(\mathcal{O}_{k'}) \cdot \pi^\mu \right) \sqcup \left( \bigsqcup_{\substack{\mu \in \Lambda_m \\ |\mu| \text{ is odd}}} GL_m(\mathcal{O}_{k'}) \cdot \pi^\mu \right), \end{aligned} \quad (1.5)$$

where  $|\mu| = \sum_{i=1}^m \mu_i$ .

**Theorem 1.3** *There are precisely two  $G$ -orbits in  $X_n$  :*

$$G \cdot x_0 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \text{ is even}}} K \cdot x_\lambda, \quad G \cdot x_1 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \text{ is odd}}} K \cdot x_\lambda.$$

where  $|\lambda| = \sum_{i=1}^n \lambda_i$ ,  $x_0 = 1_{2n+1}$  and  $x_1 = \text{Diag}(\pi, 1, \dots, 1, \pi^{-1})$ .

## §2 Spherical function $\omega(x; s)$ on $X$

We introduce a spherical function  $\omega(x; s)$  on  $X$  by Poisson transform from relative  $B$ -invariants. For a matrix  $g \in G$ , denote by  $d_i(g)$  the determinant of lower right  $i$  by  $i$  block of  $g$ . Then  $d_i(x)$ ,  $1 \leq i \leq n$  are relative  $B$ -invariants on  $X$  associated with rational characters  $\psi_i$  of  $B$ , where

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)), \quad (x \in X, p \in B). \quad (2.1)$$

We set

$$X^{op} = \{x \in X \mid d_i(x) \neq 0, 1 \leq i \leq n\}. \quad (2.2)$$

then  $X^{op}(\bar{k})$  is a Zariski open  $B(\bar{k})$ -orbit, where  $\bar{k}$  is the algebraic closure of  $k$ . For  $x \in X$  and  $s \in \mathbb{C}^n$ , we consider the integral

$$\omega(x; s) = \int_K |\mathbf{d}(k \cdot x)|^s dk, \quad |\mathbf{d}(y)|^s = \prod_{i=1}^n |d_i(y)|^{s_i}, \quad (2.3)$$

where  $dk$  is the normalized Haar measure on  $K$ , and  $k$  runs over the set  $\{k \in K \mid k \cdot x \in X^{op}\}$ . The right hand side of (2.3) is absolutely convergent if  $\operatorname{Re}(s_i) \geq 0$ ,  $1 \leq i \leq n$ , and continued to a rational function of  $q^{s_1}, \dots, q^{s_n}$ , and we use the notation  $\omega(x; s)$  in such sense. We call  $\omega(x; s)$  a spherical function on  $X$ , since it becomes an  $\mathcal{H}(G, K)$ -common eigenfunction on  $X$  (cf. [H1, §1], or [H2, §1]). Indeed,  $\mathcal{H}(G, K)$  is a commutative  $\mathbb{C}$ -algebra spanned by all the characteristic functions of double cosets  $KgK$ ,  $g \in G$  by definition, and we see

$$\begin{aligned} (f * \omega(\cdot; s))(x) &= \int_G f(g) \omega(g^{-1} \cdot x; s) dg \\ &= \lambda_s(f) \omega(x; s), \quad (f \in \mathcal{H}(G, K)), \end{aligned} \quad (2.4)$$

where  $dg$  is the Haar measure on  $G$  normalized by  $\int_K dg = 1$ , and  $\lambda_s$  is the  $\mathbb{C}$ -algebra homomorphism defined by

$$\begin{aligned} \lambda_s : \mathcal{H}(G, K) &\longrightarrow \mathbb{C}(q^{s_1}, \dots, q^{s_n}), \\ f &\longmapsto \int_B f(p) |\psi(p)|^{-s} \delta(p) dp. \end{aligned}$$

Here  $|\psi(p)|^{-s} = \prod_{i=1}^n |\psi_i(p)|^{-s_i}$ ,  $dp$  is the left-invariant measure on  $P$  such that  $\int_{P \cap K} dp = 1$  and  $\delta(p)$  is the modulus character of  $dp$  ( $d(pq) = \delta(q)^{-1} dp$ ).

We introduce a new variable  $z$  which is related to  $s$  by

$$s_i = -z_i + z_{i+1} - 1 + \frac{\pi\sqrt{-1}}{\log q} \quad (1 \leq i \leq n-1), \quad s_n = -z_n - 1 + \frac{\pi\sqrt{-1}}{2\log q} \quad (2.5)$$

and write  $\omega(x; z) = \omega(x; s)$ . We see

$$|\psi(p)|^s = (-1)^{v_\pi(p_1 \cdots p_n)} \prod_{i=1}^n |N_{k'/k}(p_i)|^{z_i} \delta^{\frac{1}{2}}(p), \quad p \in B, \quad (2.6)$$



where  $p_i$  is the  $i$ -th diagonal entry of  $p$ ,  $1 \leq i \leq n$ . The Weyl group  $W$  of  $G$  with respect to the maximal  $k$ -split torus in  $B$  acts on rational characters of  $B$  as usual (i.e.,  $\sigma(\psi)(b) = \psi(n_\sigma^{-1}bn_\sigma)$  by taking a representative  $n_\sigma$  of  $\sigma$ ), so  $W$  acts on  $z \in \mathbb{C}^n$  and on  $s \in \mathbb{C}^n$  as well. We will determine the functional equations of  $\omega(x; s)$  with respect to this Weyl group action. The group  $W$  is isomorphic to  $S_n \ltimes C_2^n$ ,  $S_n$  acts on  $z$  by permutation of indices, and  $W$  is generated by  $S_n$  and  $\tau : (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, -z_n)$ . Keeping the relation (2.5), we also write  $\lambda_z(f) = \lambda_s(f)$ . Then the  $\mathbb{C}$ -algebra map  $\lambda_z$  is an isomorphism (the Satake isomorphism)

$$\lambda_z : \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W, \quad (2.7)$$

where the ring of the right hand side is the invariant subring of the Laurent polynomial ring  $\mathbb{C}[q^{2z_1}, q^{-2z_1}, \dots, q^{2z_n}, q^{-2z_n}]$  by  $W$ .

By the embedding

$$K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K, \quad h \longmapsto \tilde{h} = \begin{pmatrix} jh^{*-1}j & & \\ & 1 & \\ & & h \end{pmatrix},$$

we obtain

$$\omega(x; s) = \int_K \zeta_*^{(h)}(D(k \cdot x); s) dk,$$

where  $D(y)$  is a spherical function on the space  $\mathcal{H}_n(k')$  defined by

$$\zeta_*^{(h)}(y; s) = \int_{K_0} |\mathbf{d}(h \cdot y)|^s dh, \quad (h \cdot y = hyh^*),$$

and we see  $G_1(z) \cdot \zeta_*^{(h)}(y; s)$  is holomorphic for  $z \in \mathbb{C}^n$  and  $S_n$ -invariant (cf. [H1, §2]). Thus we obtain the following.

**Theorem 2.1** *The function  $G_1(z) \cdot \omega(x; s)$  is invariant under the action of  $S_n$  on  $z$ , where*

$$G_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}}. \quad (2.8)$$

As for  $\tau \in W$ , first we calculate explicitly the spherical function  $\omega^{(1)}(x; s)$  of  $n = 1$  and read

$$\frac{1 + q^{2z}}{1 - q^{-1+2z}} \cdot \omega^{(1)}(x; z) \in \mathbb{C}[q^z + q^{-z}], \quad \omega^{(1)}(x; z) = \frac{1 - q^{-1+2z}}{q^{2z} - q^{-1}} \cdot \omega^{(1)}(x; \tau(z)). \quad (2.9)$$

Then, for  $n \geq 2$ , by using the embedding

$$K_1 = U(j_3)(\mathcal{O}_{k'}) \hookrightarrow K = K_n, \quad h \longmapsto \tilde{h} = \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix},$$

we show the identity

$$\omega(x; s) = \int_K \prod_{i=1}^{n-2} |d_i(k \cdot x)|^{s_i} \cdot |d_{n-1}(k \cdot x)|^{s_{n-1}+s_n} \cdot \omega^{(1)}(D_1(k \cdot x); s_n) dk, \quad (2.10)$$

where,  $D_1(y) \in X_1$  for  $y \in X^\sigma$  and  $D_1(y) = \text{Diag}(y_n, y_0, y_n^{-1})$  if  $y$  is diagonal with diagonal  $n$ -th entry  $y_n$  and  $(n+1)$ -entry  $y_0$ . By (2.9) and (2.10), we obtain the following.

**Theorem 2.2** *For general size  $n$ , the spherical function satisfies the functional equation*

$$\omega(x; z) = \frac{1 - q^{-1+2z_n}}{q^{2z_n} - q^{-1}} \omega(x; \tau(z)),$$

where  $\tau(z) = (z_1, \dots, z_{n-1}, -z_n)$ .

We prepare some notation. Set

$$\begin{aligned} \Sigma &= \{\pm e_i \pm e_j, \pm 2e_i \mid 1 \leq i, j \leq n, i \neq j\}, \quad \Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+, \\ \Sigma_s^+ &= \{e_i + e_j, e_i - e_j \mid 1 \leq i < j \leq n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\}, \end{aligned}$$

where  $e_i$  is the  $i$ -th unit vector in  $\mathbb{Z}^n$ ,  $1 \leq i \leq n$ . We note here that  $\Sigma \cup \{\pm e_i \mid 1 \leq i \leq n\}$  is the set of roots of  $G$ . We consider the pairing

$$\langle t, z \rangle = \sum_{i=1}^n t_i z_i, \quad (t, z) \in \mathbb{Z}^n \times \mathbb{C}^n,$$

which satisfies

$$\langle \alpha, z \rangle = \langle \sigma(\alpha), \sigma(z) \rangle, \quad (\alpha \in \Sigma, z \in \mathbb{C}^n, \sigma \in W).$$

Then the following two theorems are proved in the same way as in the even case, based on Theorem 2.1 and Theorem 2.2.

**Theorem 2.3** *The spherical function  $\omega(x; z)$  satisfies the following functional equation*

$$\omega(x; z) = \Gamma_\sigma(z) \cdot \omega(x; \sigma(z)), \quad (2.11)$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma^+(\sigma)} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}}, \quad \Sigma^+(\sigma) = \{\alpha \in \Sigma^+ \mid -\sigma(\alpha) \in \Sigma^+\}.$$

**Theorem 2.4** *The function  $G(z) \cdot \omega(x; z)$  is holomorphic on  $\mathbb{C}^n$  and  $W$ -invariant. In particular it is an element in  $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$ , where*

$$G(z) = \prod_{\alpha \in \Sigma^+} \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle - 1}}.$$

### §3 The explicit formula for $\omega(x; z)$

In order to determine the explicit formula of  $\omega(x; z)$ , it suffices to give the explicit formula for each representative of  $K$ -orbits, hence for  $x_\lambda, \lambda \in \Lambda_n^+$  by Theorem 1.1.

**Theorem 3.1** *For  $\lambda \in \Lambda_n^+$ , one has the explicit formula*

$$\omega(x_\lambda; z) = \frac{(1+q^{-1})(1-q^{-2})^n}{w_{2n+1}(-q^{-1})} \cdot \frac{1}{G(z)} \cdot q^{\langle \lambda, z_0 \rangle} \cdot Q_\lambda(z),$$

where  $G(z)$  is given in Theorem 2.4,  $z_0 \in \mathbb{C}^n$  is the value in  $z$ -variable corresponding to  $0 \in \mathbb{C}^n$  in  $s$ -variable,

$$z_{0,i} = -(n-i+1) + (n-i+\frac{1}{2}) \frac{\pi\sqrt{-1}}{\log q}, \quad 1 \leq i \leq n, \quad (3.1)$$

$$w_m(t) = \prod_{i=1}^m (1-t^i),$$

$$Q_\lambda(z) = Q_\lambda(z; -q^{-1}, -q^{-2}), \quad Q_\lambda(z; t) = \sum_{\sigma \in W} \sigma(q^{-\langle \lambda, z \rangle} c(z, t)),$$

$$c(z; t) = \prod_{\alpha \in \Sigma^+} \frac{1 - t_\alpha q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}}, \quad t_\alpha = \begin{cases} t_s = -q^{-1} & \text{if } \alpha \in \Sigma_s^+ \\ t_\ell = -q^{-2} & \text{if } \alpha \in \Sigma_\ell^+. \end{cases} \quad (3.2)$$

We prove Theorem 3.1 by using a general expression formula of spherical functions given in [H2, §2] based on the functional equations of spherical functions and data of the group  $G$  under certain condition. After checking the condition, which is rather troublesome, we use the results in §2, and formulate the formula.

**Remark 3.2** We see the main part  $Q_\lambda(z)$  of  $\omega(x; z)$  is contained in  $\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$  by Theorem 2.4, and

$$P_\lambda(z; t_s, t_\ell) = \frac{1}{W_\lambda(t_s, t_\ell)} \cdot Q_\lambda(z; t_s, t_\ell) \quad (3.3)$$

is a specialization of Hall-Littlewood polynomial of type  $C_n$ , where  $W_\lambda(t_s, t_\ell)$  is the Poincaré polynomial of the stabilizer subgroup  $W_\lambda$  of  $W$  at  $\lambda$ , and

$$W_\lambda(-q^{-1}, -q^{-2}) = \frac{\widetilde{w}_\lambda(-q^{-1})}{(1+q^{-1})^{n+1}}, \quad (3.4)$$

$$\widetilde{w}_\lambda(t) = \begin{cases} w_{m_0(\lambda)+1}(t) \cdot \prod_{\ell \geq 0} w_{m_\ell(\lambda)}(t) & \text{if } m_0 > 0 \\ \prod_{\ell \geq 1} w_{m_\ell(\lambda)}(t) & \text{if } m_0 = 0, \end{cases} \quad m_\ell(\lambda) = \#\{i \mid \lambda_i = \ell\}.$$

Using  $P_\lambda(z) = P_\lambda(z; -q^{-1}, -q^{-2})$  we have

$$\omega(x_\lambda; z) = \frac{(1-q^{-1})^n}{w_{2n+1}(-q^{-1})} \cdot \frac{1}{G(z)} \cdot \widetilde{w}_\lambda(-q^{-1}) \cdot q^{\langle \lambda, z_0 \rangle} \cdot P_\lambda(z), \quad (\lambda \in \Lambda_n^+). \quad (3.5)$$

It is known (cf. [M2], [HK, Appendix B]) that the set  $\{P_\lambda(z; t_s, t_\ell) \mid \lambda \in \Lambda_n^+\}$  forms an orthogonal  $\mathbb{C}$ -basis for  $\mathcal{R}$ , in particular  $P_0(z) = 1$  (cf. (4.4) and (4.6)). In the present case, the root system of the group  $G = U(j_{2n+1})$  is of type  $BC_n$ , but we can write the explicit formula for  $\omega(x; z)$  as above by using  $P_\lambda(z; t_s, t_\ell)$  of type  $C_n$ . We need a different specialization from the case of unitary hermitian forms of even size, which is a homogeneous space of the group  $U(j_{2n})$  of type  $C_n$  and  $(t_s, t_\ell) = (-q^{-1}, q^{-1})$  (cf. [HK]).

We have the following immediately from (3.5).

**Corollary 3.3** *For  $x_0 = 1_{2n+1}$ , one has*

$$\omega(1_{2n+1}; z) = \frac{(1 - q^{-1})^n w_n(-q^{-1}) w_{n+1}(-q^{-1})}{w_{2n+1}(-q^{-1})} \times \frac{1}{G(z)}.$$

**Remark 3.4** We give an interpretation of the constant  $z_0$ . For  $v \in \mathbb{Z}^n$ , let

$$t^{\text{ht}(v)} = \prod_{\beta \in \Sigma_s^+} t_s^{\langle v, \beta^\vee \rangle / 2} \prod_{\beta \in \Sigma_\ell^+} t_\ell^{\langle v, \beta^\vee \rangle / 2}, \quad (3.6)$$

where  $\beta^\vee = 2\beta / \langle \beta, \beta \rangle$ . Then for  $v = \alpha \in \Sigma$ , this coincides with the generalization of the heights of roots [M1]. On the other hand, (3.6) can be rewritten as

$$t^{\text{ht}(v)} = q^{\langle v, z_0 \rangle},$$

where  $z_0$  is given by (3.1). Thus  $z_0$  can be regarded as a generalization of the dual Weyl vector.

From this viewpoint, the constant  $z_0$  in the even case is calculated as

$$z_{0,i} = -(n - i + \frac{1}{2}) + (n - i) \frac{\pi \sqrt{-1}}{\log q}, \quad 1 \leq i \leq n,$$

which corresponds to the change of variables

$$s_n = -z_n - \frac{1}{2} \quad (3.7)$$

in (0.3) with the same  $s_i$  ( $1 \leq i \leq n-1$ ) as before. This modification causes only the sign changes of  $\omega(x_\lambda; z)$ , that is,  $\omega(x_\lambda; z)$  with (3.7) is the multiple by  $(-1)^{|\lambda|}$  of the original  $\omega(x_\lambda; z)$ . In other words, on  $G \cdot x_0$  the both coincide and on  $G \cdot x_1$  the difference is the multiple by  $-1$ .

## §4 Spherical Fourier transform and Plancherel formula on $\mathcal{S}(K \backslash X)$

We modify the spherical function by

$$\Psi(x; z) = \omega(x; z) / \omega(1_{2n}; z) \in \mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W, \quad (4.1)$$

and define the spherical Fourier transform on the Schwartz space

$$\mathcal{S}(K \backslash X) = \{ \varphi : X \rightarrow \mathbb{C} \mid \text{left } K\text{-invariant, compactly supported} \},$$

by

$$\begin{aligned} F : \mathcal{S}(K \backslash X) &\longrightarrow \mathcal{R} \\ \varphi &\longmapsto F(\varphi)(z) = \int_X \varphi(x) \Psi(x; z) dx, \end{aligned} \quad (4.2)$$

where  $dx$  is a  $G$ -invariant measure on  $X$ . The existence of a  $G$ -invariant measure is assured by the fact  $X$  is a union of two  $G$ -orbits and  $G$  is reductive, and we fix the normalization afterwards. Then, for each characteristic function  $ch_\lambda$  of  $K \cdot x_\lambda$ ,  $\lambda \in \Lambda_n^+$ , we have

$$F(ch_\lambda)(z) = q^{\langle \lambda, z_0 \rangle} \frac{\widetilde{w}_\lambda(-q^{-1})}{\widetilde{w}_0(-q^{-1})} \cdot v(K \cdot x_\lambda) P_\lambda(z), \quad (4.3)$$

where  $\widetilde{w}_\lambda(-q^{-1})$  is defined in Remark 3.2, and  $v(K \cdot x_\lambda)$  is the volume of  $K \cdot x_\lambda$  with respect to  $dx$ . On the other hand, we regard  $\mathcal{R}$  as an  $\mathcal{H}(G, K)$ -module through the Satake isomorphism  $\lambda_z$  (cf. (2.7)). Then we have the following.

**Theorem 4.1** *The spherical Fourier transform  $F$  gives an  $\mathcal{H}(G, K)$ -module isomorphism*

$$\mathcal{S}(K \backslash X) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W (= \mathcal{R}),$$

where  $\mathcal{R}$  is regarded as  $\mathcal{H}(G, K)$ -module via  $\lambda_z$ . Especially,  $\mathcal{S}(K \backslash X)$  is a free  $\mathcal{H}(G, K)$ -module of rank  $2^n$ .

**Corollary 4.2** *All the spherical functions on  $X$  are parametrized by eigenvalues*

$z \in \left( \mathbb{C} / \frac{2\pi\sqrt{-1}}{\log q} \mathbb{Z} \right)^n / W$  through  $\mathcal{H}(G, K) \rightarrow \mathbb{C}$ ,  $f \mapsto \lambda_z(f)$ . The set  $\{ \Psi(x; z + u) \mid u \in \{0, \pi\sqrt{-1}/\log q\}^n \}$  forms a basis of the space of spherical functions on  $X$  corresponding to  $z$ .

We introduce an inner product  $\langle \cdot, \cdot \rangle_{\mathcal{R}}$  on  $\mathcal{R}$  by

$$\langle P, Q \rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z) \overline{Q(z)} d\mu(z), \quad P, Q \in \mathcal{R}. \quad (4.4)$$

Here

$$\mathfrak{a}^* = \left\{ \sqrt{-1} \left( \mathbb{R} / \frac{2\pi}{\log q} \mathbb{Z} \right) \right\}^n, \quad d\mu = \frac{1}{n!2^n} \cdot \frac{w_n(-q^{-1})w_{n+1}(-q^{-1})}{(1+q^{-1})^{n+1}} \cdot \frac{1}{|c(z)|^2} dz, \quad (4.5)$$

where  $dz$  is the Haar measure on  $\mathfrak{a}^*$  with  $\int_{\mathfrak{a}^*} dz = 1$  and  $c(z) = c(z; -q^{-1}, -q^{-2})$  is defined in Theorem 3.1. Then, it is known

$$\langle P_\lambda, P_\mu \rangle_{\mathcal{R}} = \langle P_\mu, P_\lambda \rangle_{\mathcal{R}} = \delta_{\lambda, \mu} \cdot \frac{\widetilde{w}_0(-q^{-1})}{\widetilde{w}_\lambda(-q^{-1})}, \quad (\lambda, \mu \in \Lambda_n^+). \quad (4.6)$$

On the other hand, one has

$$\frac{v(K \cdot x_\lambda)}{v(K \cdot x_\mu)} = \frac{q^{2\langle \mu, z_0 \rangle} \widetilde{w}_\mu(-q^{-1})}{q^{2\langle \lambda, z_0 \rangle} \widetilde{w}_\lambda(-q^{-1})}, \quad (\lambda, \mu \in \Lambda_n^+, |\lambda| \equiv |\mu| \pmod{2}). \quad (4.7)$$

**Theorem 4.3 (Plancherel formula on  $\mathcal{S}(K \backslash X)$ )** *Let  $d\mu$  be the measure defined by (4.5). By the normalization of  $G$ -invariant measure  $dx$  such that*

$$v(K \cdot x_\lambda) = q^{-2\langle \lambda, z_0 \rangle} \frac{\widetilde{w}_0(-q^{-1})}{\widetilde{w}_\lambda(-q^{-1})}, \quad \lambda \in \Lambda_n^+, \quad (4.8)$$

*one has for any  $\varphi, \psi \in \mathcal{S}(K \backslash X)$*

$$\int_X \varphi(x) \overline{\psi(x)} dx = \int_{\mathfrak{a}^*} F(\varphi)(z) \overline{F(\psi)(z)} d\mu(z). \quad (4.9)$$

*Outline of a proof.* We recall the  $G$ -orbit decomposition (Theorem 1.3) and normalize  $dx$  on each  $G$ -orbit as

$$v(K \cdot x_\lambda) = \begin{cases} 1 & \text{for } \lambda = \mathbf{0}, \\ q^{-2n \frac{(1-(-q^{-1})^n)(1-(-q^{-1})^{n+1})}{1+q^{-1}}} & \text{for } \lambda = (1, 0, \dots, 0), \end{cases}$$

then the identity (4.8) follows from this and (4.7). Then, for any  $\lambda, \mu \in \Lambda_n^+$ , we see by (4.3), (4.6), and (4.8)

$$\int_X ch_\lambda(x) \overline{ch_\mu(x)} dx = \delta_{\lambda, \mu} q^{-2\langle \lambda, z_0 \rangle} \frac{\widetilde{w}_0(-q^{-1})}{\widetilde{w}_\lambda(-q^{-1})} = \int_{\mathfrak{a}^*} F(ch_\lambda)(z) \overline{F(ch_\mu)(z)} d\mu(z).$$

Since  $\{ch_\lambda \mid \lambda \in \Lambda_n^+\}$  spans  $\mathcal{S}(K \backslash X)$ , we conclude the identity (4.9). ■

The next corollary is an easy consequence of Theorem 4.3.

**Corollary 4.4 (Inversion formula)** *For any  $\varphi \in \mathcal{S}(K \backslash X)$ ,*

$$\varphi(x) = \int_{\mathfrak{a}^*} F(\varphi)(z) \Psi(x; z) d\mu(z), \quad x \in X.$$

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